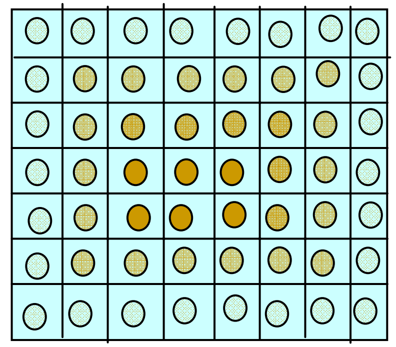
多重積分
multiple integrals

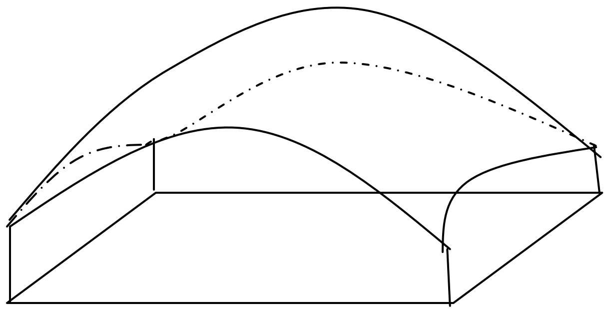
長方形の領域の二重積分
double integrals over rectangles

- 微小な面積に重みをつけて和を取る
- 重みを表す関数のグラフの下側の体積
- 面積素片を0にする極限、無限の和



$$S_n = \sum f(x_k, y_k)\Delta A_k = \sum f(x_k, y_k)\Delta x \Delta y$$

$$\lim S_n = \iint_R f(x, y)dA = \iint_R f(x, y)dxdy$$



二重積分の性質

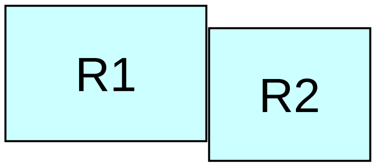
$$\iint_R kf(x,y)dxdy = k\iint_R f(x,y)dxdy$$

$$\iint_R \left(f(x,y) \pm g(x,y) \right) dxdy = \iint_R f(x,y)dxdy \pm \iint_R g(x,y)dxdy$$

$$\forall (x,y) \in R \text{で} \ f(x,y) \geq 0 \Rightarrow \iint_R f(x,y)dxdy \geq 0$$

$$\forall (x,y) \in R \text{で} \ f(x,y) \geq g(x,y) \Rightarrow \iint_R f(x,y)dxdy \geq \iint_R g(x,y)dxdy$$

$$\iint_{R_1+R_2} f(x,y)dxdy = \iint_{R_1} f(x,y)dxdy + \iint_{R_2} f(x,y)dxdy$$



R1とR2は重ならない

体積積分を薄板を集めて計算

$$A(x)dx = \left(\int_{y=y_{\min}}^{y=y_{\max}} f(x,y)dy \right) dx$$

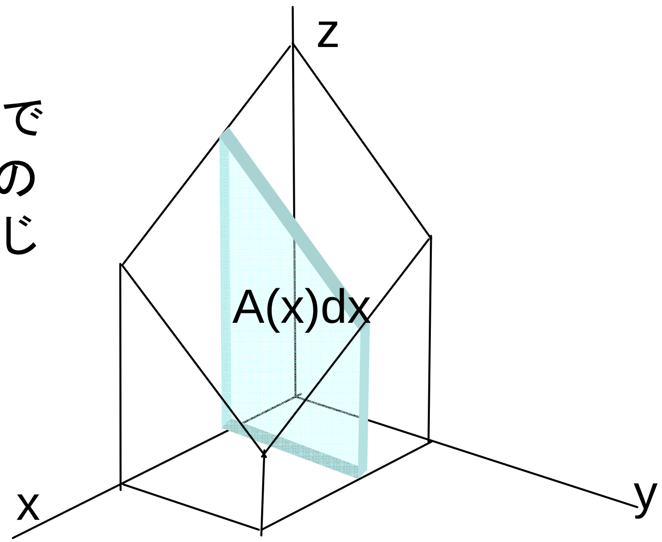
A(x):xを通る板の面積
A(x)dx:体積

$$\int_{x=x_{\min}}^{x=x_{\max}} A(x)dx = \int_{x=x_{\min}}^{x=x_{\max}} \left(\int_{y=y_{\min}}^{y=y_{\max}} f(x,y)dy \right) dx = \int_{x=x_{\min}}^{x=x_{\max}} \int_{y=y_{\min}}^{y=y_{\max}} f(x,y)dydx$$

$$\int_{y=y_{\min}}^{y=y_{\max}} \int_{x=x_{\min}}^{x=x_{\max}} f(x,y)dxdy$$

$$= \int_{x=x_{\min}}^{x=x_{\max}} \int_{y=y_{\min}}^{y=y_{\max}} f(x,y)dydx$$

関数 f が領域Rで連続なら、積分の順序によらず同じ値になる



例題

$$f(x,y)=1-6x^2y,\qquad R:\ 0\le x\le 2,\ -1\le y\le 1$$

$$\iint_R f(x,y)dxdy=\int_{-1}^{1}\left(\int_0^2\left(1-6x^2y\right)dx\right)dy=\int_{-1}^{1}\left[x-2x^3y\right]_{x=0}^{2}dy$$

$$=\int_{-1}^{1}(2-16y)dy=\left[2y-8y^2\right]_{-1}^{1}=4$$

$$\int_0^2\left(\int_{-1}^{1}\left(1-6x^2y\right)dy\right)dx=\int_0^2\left[y-3x^2y^2\right]_{y=-1}^{1}dx$$

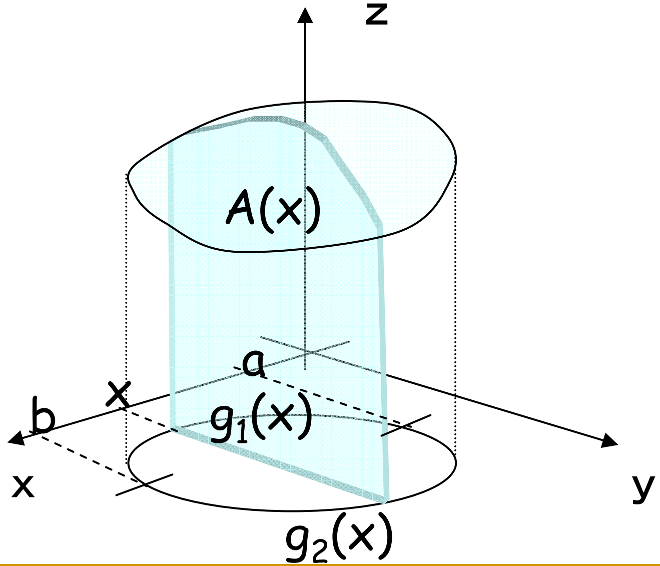
$$=\int_0^2\left[(1-3x^2)-(-1-3x^2)\right]dx=\int_0^2 2dx=4$$

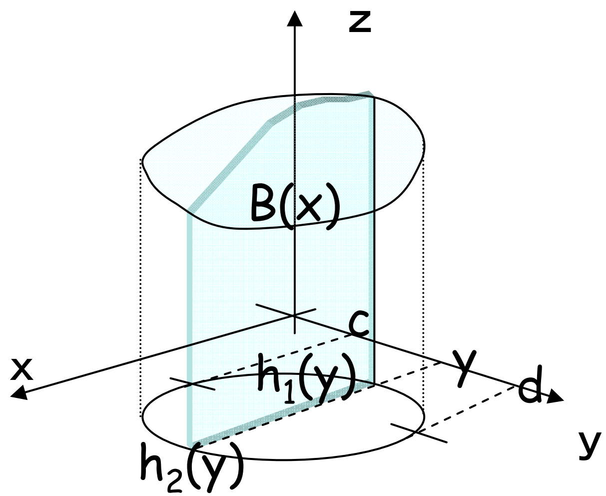
二重積分
長方形領域→任意形状の領域

$$A(x) = \int_{y=g_1(x)}^{y=g_2(x)} f(x, y)dy$$

$$B(x) = \int_{x=h_1(y)}^{x=h_2(y)} f(x, y)dx$$

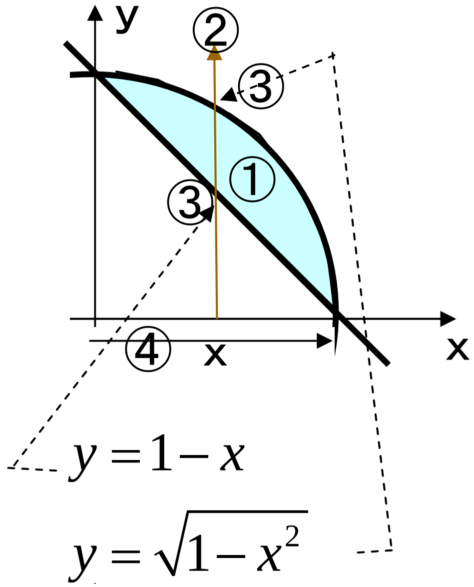
$$\int_a^b A(x)dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)dydx \quad = \quad \int_c^d B(x)dx = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)dxdy$$





例題 $f(x,y),\ \ R: x+y=1$ と $x^2+y^2=1$で囲まれた領域

1．領域のグラフを描く
2．x軸上の点xを通過するy軸に平行な直線をひく
3．領域の境界線との交点を求める
4．xが動く領域を求める



$$0 \le x \le 1$$

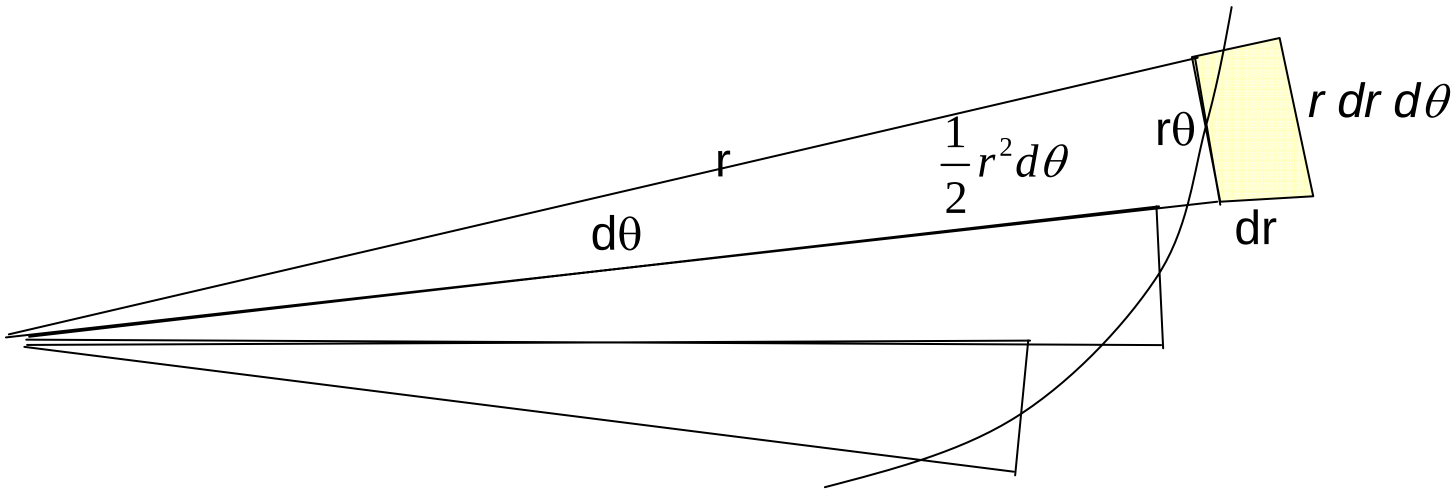
$$\int_{y=1-x}^{y=\sqrt{1-x^2}} f(x,y)dy$$

$$\int_0^1 \left(\int_{y=1-x}^{y=\sqrt{1-x^2}} f(x,y)dy \right) dx$$

極座標による面積の計算

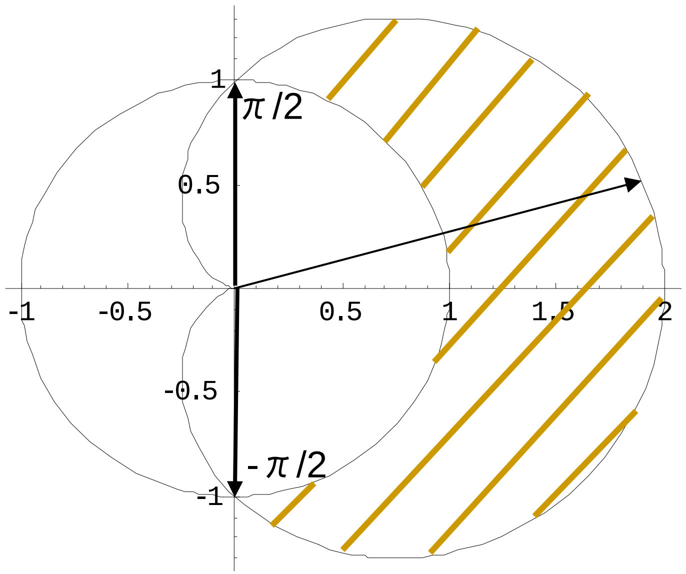
扇型 $:\frac{1}{2}r^2 d\theta$

面積素片 $: dS = \frac{1}{2}(r+dr)^2 d\theta - \frac{1}{2}r^2 d\theta = rdrd\theta$



例題

$$\int_{-\pi/2}^{\pi/2}\int_{1}^{1+\cos\theta} f(r,\theta)\,r\,dr\,d\theta$$



$r = 1$

$r = 1 + \cos\theta$

積分変数の変換
ヤコビアン

$$x = g(u,v),\ y = h(u,v)$$

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

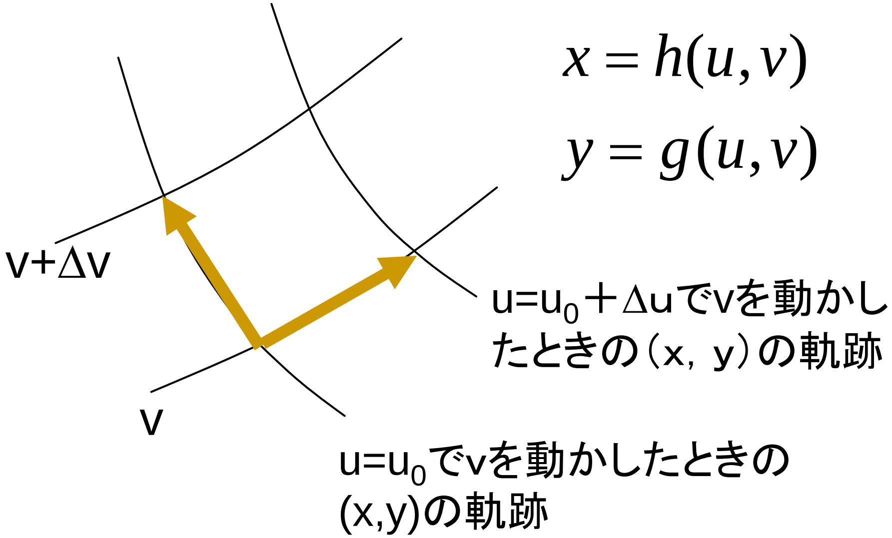
$$\iint_R f(x,y)\,dxdy = \iint_R f(g(u,v),h(u,v))J(u,v)\,dudv$$

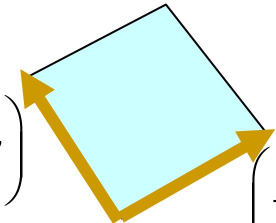
ヤコビアン

- 新しい変数u,vを微小に変化させたとき，xy面上にどれだけの面積が生じるか

$$x = h(u,v)$$

$$y = g(u,v)$$





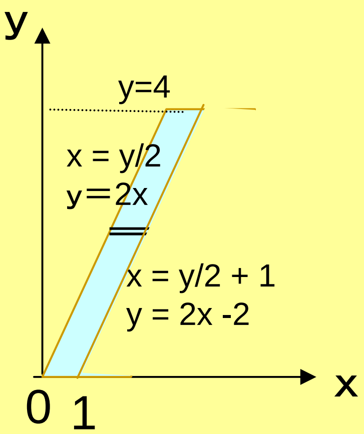
$$\begin{vmatrix} \frac{\partial x}{\partial u}\Delta u & \frac{\partial y}{\partial u}\Delta u \\ \frac{\partial x}{\partial v}\Delta v & \frac{\partial y}{\partial v}\Delta v \end{vmatrix} = \frac{\partial x}{\partial u}\Delta u \frac{\partial y}{\partial v}\Delta v - \frac{\partial y}{\partial u}\Delta u \frac{\partial x}{\partial v}\Delta v = \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} \right)\Delta u \Delta v$$

例：

$$\int_0^4 \int_{x=\frac{y}{2}}^{x=\frac{y}{2}+1} \frac{2x-y}{2}\,dxdy$$ を $$u=\frac{2x-y}{2}, v=\frac{y}{2}$$ とおいて計算せよ。

$$x=u+v, y=2v$$

$$J(u,v)=\begin{vmatrix} \frac{\partial(u+v)}{\partial u} & \frac{\partial(u+v)}{\partial v} \\ \frac{\partial(2v)}{\partial u} & \frac{\partial(2v)}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2$$



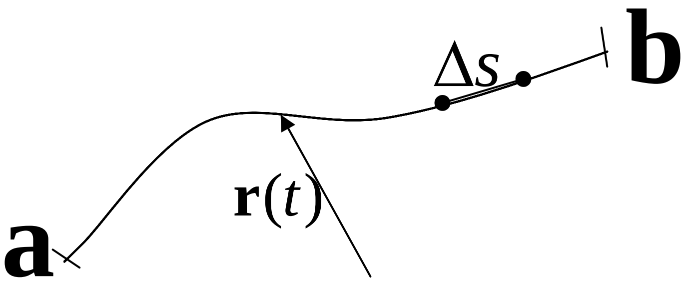
よって

$$\int_0^4 \int_{x=\frac{y}{2}}^{x=\frac{y}{2}+1} \frac{2x-y}{2}\,dxdy = \int_{v=0}^{v=2}\int_{u=0}^{u=1} uJ(u,v)\,dudv = \int_{v=0}^{v=2}\int_{u=0}^{u=1} 2u\,dudv = \int_0^2 \left[u^2\right]_0^1 dv = \int_0^2 dv = 2$$

線積分　line integral

曲線にそって移動しながら積分する

- 曲線を小区間に分割
  - 小区間の長さ
  - Δs:経路により変わる
- 曲線上の点　（t　で指定）
  - **r**(t)
  - 関数もtを変数とする
- 点aからbまで、曲線Cに沿うfの線積分
  - tによる積分



$$\int_C f(x,y,z)ds = \lim \sum_{i=1}^{N} f(x_i, y_i, z_i)\Delta s_i$$

$$C : \mathbf{r}(t) = x(t)\mathbf{e}_x + y(t)\mathbf{e}_y + z(t)\mathbf{e}_z$$

$$d\mathbf{r} = dx\mathbf{e}_x + dy\mathbf{e}_y + dz\mathbf{e}_z$$

$$ds = |d\mathbf{r}| = \sqrt{dx^2 + dy^2 + dz^2}$$

$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = |\mathbf{v}|dt$$

$$f(\mathbf{r}) = f(x,y,z) = f(x(t), y(t), z(t)) = f(t)$$

$$\int_C f(x,y,z)ds = \int_{t_a \to t_b} f(t)|\mathbf{v}|ds$$

パラメータ表示

例

$$f(x,y,z) = x - 3y^2 + z$$

C：線分、原点と (1, 1, 1) を結ぶ

$$\mathbf{r}(t) = t\mathbf{e}_x + t\mathbf{e}_y + t\mathbf{e}_z, 0 \le t \le 1$$

$$|\mathbf{v}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$\int_C f(x,y,z)ds = \int_C f(x,y,z)\Big|_{\substack{x=t\\y=t\\z=t}} \sqrt{3}dt$$

$$= \int_0^1 \sqrt{3}\left(t - 3t^2 + t\right)dt$$

$$= \sqrt{3}\int_0^1 \left(2t - 3t^2\right)dt$$

$$= \sqrt{3}\left[t^2 - t^3\right]_0^1 = 0$$

加算性

- 曲線をいくつかの区間にわけ，それぞれが重ならないなら，区間ごとの線積分の和となる

$$\int_{C=C_1+C_2+\cdots} f(\mathbf{r})ds = \int_{C_1} f(\mathbf{r})ds + \int_{C_2} f(\mathbf{r})ds + \cdots$$

例

$$f(x,y,z) = x - 3y^2 + z$$

C : 折れ線

$$(0,0,0) \underset{dx}{\rightarrow} (1,0,0) \underset{dy}{\rightarrow} (1,1,0) \underset{dz}{\rightarrow} (1,1,1)$$

$$\int_{(0,0,0)}^{(1,0,0)} f(x,y,z)ds = \int_{x=0}^{1} f(x,0,0)dx = \int_0^1 xdx = \frac{1}{2}$$

$$\int_{(1,0,0)}^{(1,1,0)} f(x,y,z)ds = \int_{y=0}^{1} f(1,y,0)dy = \int_0^1 \left(1 - 3y^2\right)dy$$

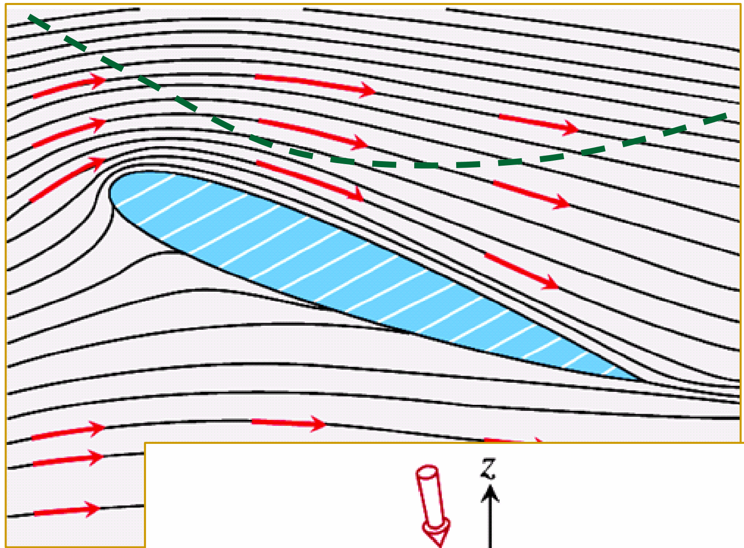
$$= \left[y - y^3 \right]_0^1 = 1 - 1 = 0$$

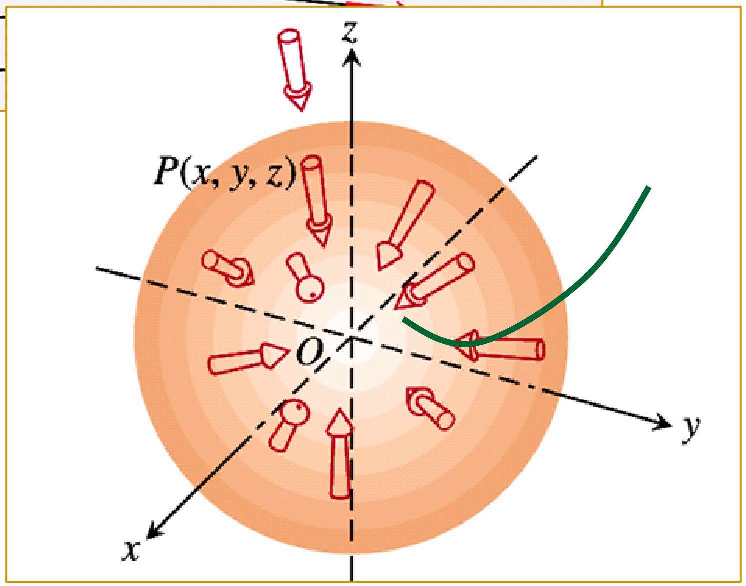
$$\int_{(1,1,0)}^{(1,1,1)} f(x,y,z)ds = \int_{z=0}^{1} f(1,1,0)dz = \int_0^1 \left(1 - 3 + z\right)dz$$

$$= \left[-2z + \frac{1}{2}z^2 \right]_0^1 = -2 + \frac{1}{2}$$

$$\int_{C'} f(x,y,z)ds = \frac{1}{2} + 0 - 2 + \frac{1}{2} = -1$$

ベクトル場の線積分





$$\mathbf{F}(\mathbf{r}) = P(\mathbf{r})\mathbf{e}_x + Q(\mathbf{r})\mathbf{e}_y + R(\mathbf{r})\mathbf{e}_z$$

$$d\mathbf{r} = dx\mathbf{e}_x + dy\mathbf{e}_y + dz\mathbf{e}_z$$

$$\mathbf{F}(\mathbf{r})\cdot d\mathbf{r} = P(\mathbf{r})dx + Q(\mathbf{r})dy + R(\mathbf{r})dz$$

$$\int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r} = \int_C \mathbf{F}(\mathbf{r})\cdot\frac{d\mathbf{r}}{dt}dt$$

$$= \int_C \left(Pdx + Qdy + Rdz\right)$$

$$= \int_C \left(P\frac{dx}{dt}dt + Q\frac{dy}{dt}dt + R\frac{dz}{dt}dt\right)$$

$$= \int_C P\frac{dx}{dt}dt + \int_C Q\frac{dy}{dt}dt + \int_C R\frac{dz}{dt}dt$$

経路のとりかたによらない場合(2D)

- 被積分関数が何らかの関数の全微分のとき
- テストの方法
- U:ポテンシャル
- F= -∇U:保存力

$$\int_c Pdx + Qdy$$

$$P = \frac{\partial U}{\partial x}, Q = \frac{\partial U}{\partial y}$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Leftrightarrow \frac{\partial^2 U}{\partial y \partial x} = \frac{\partial^2 U}{\partial x \partial y},$$

$$U(x, y)$$

$$dU = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy = \nabla U \cdot d\mathbf{r}$$

$$\int_A^B \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy = \int_A^B dU = U(B) - U(A)$$

経路のとりかたによらない場合(3D)

- 被積分関数が何らかの関数の全微分のとき
- テストの方法
- U:ポテンシャル
- F= -∇U:保存力

$$\int_c Pdx + Qdy + Rdz$$

$$P = \frac{\partial U}{\partial x}, Q = \frac{\partial U}{\partial y}, R = \frac{\partial U}{\partial z}$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Leftrightarrow \frac{\partial^2 U}{\partial y \partial x} = \frac{\partial^2 U}{\partial x \partial y},$$

$$\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$$

$$U(x, y, z)$$

$$dU = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy + \frac{\partial U}{\partial z} dz = \nabla U \cdot d\mathbf{r}$$

$$\int_A^B \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy + \frac{\partial U}{\partial z} dz = \int_A^B dU = U(B) - U(A)$$

渦無し(rotation free)

$$\int_c Pdx + Qdy + Rdz$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$$

$$\mathbf{F} = P\mathbf{e}_x + Q\mathbf{e}_y + R\mathbf{e}_z$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right)\mathbf{e}_x + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right)\mathbf{e}_y + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\mathbf{e}_z = 0$$

例

$\mathbf{F} = (2x-3)\mathbf{e}_x - z\mathbf{e}_y + \cos z\mathbf{e}_z$

は保存力ではない。

$$\frac{\partial}{\partial x}(-z) - \frac{\partial}{\partial y}(2x-3) = 0$$

$$\frac{\partial}{\partial y}(\cos z) - \frac{\partial}{\partial z}(-z) = 1$$

$$\frac{\partial}{\partial x}(\cos z) - \frac{\partial}{\partial z}(2x-3) = 0$$

$$\int_{(1,1,1)}^{(2,3,-1)} ydx + xdy + 4dz$$

$$P(x,y,z) = y, Q(x,y,z) = x, R(x,y,z) = 4$$

$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 1-1 = 0, \frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y} = 0, \frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} = 0$$

$$df = ydx + xdy + 4dz$$

$$f = xy + 4z + const.$$

$$\int_{(1,1,1)}^{(2,3,-1)} ydx + xdy + 4dz = \left[xy + 4z\right]_{(1,1,1)}^{(2,3,-1)}$$

$$= (6-4) - (1+4) = -3$$